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Velocity and Shear Stress Distribution of Laminar Flow between Parallel Plate by Laplace Transform Approach

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Abstract: Viscosity is the property of a fluid in which an internal frictional force (viscous force) produced by intermolecular forces becomes active when the fluid is in motion and opposes the relative motion of its different layers, just as surface tension is the property of a fluid at rest. When the different layers of the fluid move at different speeds, this viscous force acts tangentially and causes a tangential (shearing) stress between the layers of the fluid in motion. In this paper, Laplace transforms approach is applied in obtaining the velocity distribution and the shear stress distribution of a unidirectional laminar flow between the stationary parallel plates as well as between the parallel plates having a relative motion by solving the differential equation describing flow characteristics of the viscous liquid via Laplace transform method. This paper shows how to use the Laplace transform to get the solution (velocity distribution and shear stress distribution) without having to find the general solution of a differential equation describing the flow characteristics equation of unidirectional laminar flow between parallel plates.

Keywords: Laminar flow, Laplace transform, parallel plates, shear stress, velocity distribution Viscous fluid.

1. Introduction

Laminar flow is the steady flow of a viscous fluid over a horizontal surface in the form of layers of different velocities that do not mix, i.e., the flow of a viscous fluid in which the fluid particles move in regular and well-defined paths. It appears that the fluid layers move over each other seamlessly. Because of the relative velocity difference between the two layers, tangential (shear) stress acts on them. Laminar flow is exemplified by seepage through soils, crude oil flow, and very viscous fluids flowing through tiny passageways. Fluid characteristics in directions perpendicular to the fluid flow direction stay unaltered in such a flow. [1-3].

1.1 Laplace Transform

Laplace transform methods have a key role to play in the modern approach to the analysis and design of engineering systems. The stimulus for developing these methods was the pioneering work of the English electrical engineer Oliver Heaviside in developing a method for the systematic solution of Ordinary Differential Equations (ODE's) with constant coefficients [4]. Research on the problem continued for many years before it was eventually recognized that an integral transform developed by the French mathematician Pierre Simon de Laplace almost a century before provided a theoretical foundation for Heaviside's work. It was also recognized that the use of this integral transform provided a more systematic alternative for investigating differential equations than the method proposed by Heaviside [5].

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The Laplace transform is an example of a class called integral transforms, and it takes a function $f(t)$ of one variable t (which we shall refer to as time) into a function $F(s)$ of another variable s (the complex frequency). The attraction of the Laplace transform is that it transforms differential equations in the t (time) domain into algebraic equations in the s (frequency) domain. Solving differential equations in the t domain therefore reduces to solving algebraic equations in the s domain. Another advantage of using the Laplace transform for solving differential equations is that initial conditions play an essential role in the transformation process, so they are automatically incorporated into the solution [6-10]. The Laplace transform is therefore an ideal tool for solving initial-value problems such as those occurring in the investigation of electrical circuits and mechanical vibrations.

The main idea of this method is to Laplace transform the constant-coefficient differential equation for $f(t)$ into a simpler algebraic equation for the Laplace-transformed function $F(s)$, solve this algebraic equation, and then transform $F(s)$ back into $f(t)$. The correct definition of the Laplace transforms and the properties that this transform satisfies makes this solution method possible. Schematically, a system may be represented as in Figure 1.

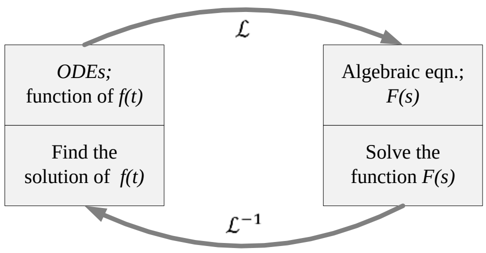

Fig. 1 – The Laplace transform operator

2. Methodology

We consider a steady and uniform laminar flow of the viscous fluid between the two flat parallel plates situated at a perpendicular distance L. Let the distance in which the fluid is flowing be represented by x and the distance which is normal to the flow of fluid and parallel to the plane of paper be represented by z such that the lower plate is situated at $z = 0$ and the upper plate is situated at $z = L$, as in Figure 2.

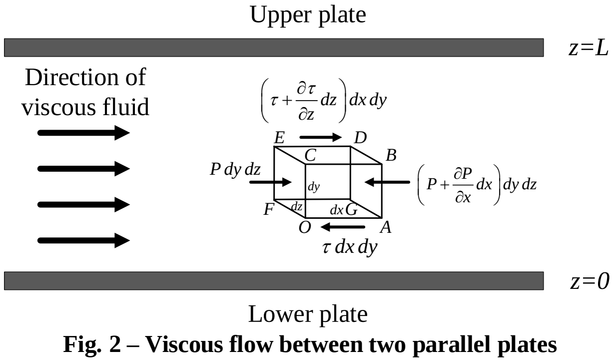

Fig. 2 – Viscous flow between two parallel plates

To derive the flow characteristics equation of motion of a viscous fluid, we consider a small fluid element of length, *dx*, breadth, *dy* and height, *dz*. Due to viscous effects, there exists a relative velocity between any two adjacent layers of the viscous fluid due to which tangential (shear) stress is set up between them. For steady and uniform flow, there will not be any shear stress on the vertical faces of the fluid element [11,12]. If the shear stress on the lower face OAGF of the fluid element is represented by τ and that on the upper face CBDE is $\left(\tau+\frac{\partial \tau}{\partial z}dz\right)$, then the tangential (shearing) force on the fluid element is $\left(\tau+\frac{\partial \tau}{\partial z}dz\right)dx\,dy-\tau\,dx\,dy=\frac{\partial \tau}{\partial z}dx\,dy\,dz$.

If the pressure intensity on the face OCEF of the fluid element is represented by P and that on the face ABDG is $\left(P+\frac{\partial P}{\partial x}dx\right)$, then pressure force on the fluid element is $P\,dy\,dz-\left(P+\frac{\partial P}{\partial x}dx\right)dy\,dz=-\frac{\partial P}{\partial x}dx\,dy\,dz$.

For steady and uniform flow, the acceleration is zero and hence the sum of the tangential (shearing) force and the pressure force in the direction of flow of the fluid which meams the resultant force in the x-direction must vanish. Thus,

$$\frac{\partial \tau}{\partial z}dx\,dy\,dz-\frac{\partial P}{\partial x}dx\,dy\,dz=0 \tag{1}$$

Since we are concentrating on pressure gradient only in the direction of flow of fluid, we can replace the partial derivatives by total derivatives. Hence,

$$\frac{\partial \tau}{\partial z}=\frac{\partial P}{\partial x} \tag{2}$$

This means that the shearing gradient in the direction normal to the flow of the viscous fluid is equal to the pressure gradient along the direction of flow of the fluid. According to Newton's law of viscosity, the value of shear stress is given by,

$$\tau=\mu\frac{dU}{dz} \tag{3}$$

where dU/dz (or be written as *U(z)* in certain case for easily arrange the report writing) represents the rate of change of velocity with respect to *z*, which means it represents the velocity gradient in the direction normal to the flow of fluid and represents the coefficient of viscosity. By putting equation (3) into the equation (2), we get,

$$\mu\frac{d^2U}{dz^2}=\frac{dP}{dx} \tag{4}$$

3. Solution of flow Characteristics Equations

We will analyse the laminar flow of viscous fluid based on following assumptions:

i. The flow is steady and incompressible, and the properties of the fluid do not vary in the directions normal to the direction of flow of the fluid.

ii. There are no end effects of the surfaces on the viscous fluid.
iii. There is a uniform effective pressure gradient in the direction of flow of fluid i.e., is a constant in the *x*–direction.
iv. There is no relative velocity of the fluid with respect to the surfaces of the plates.

Now taking Laplace Transform of equation (4), we get,

$$\mu\left[s^2U(z)-sU(0)-U'(0)\right]=\frac{1}{s}\frac{dP}{dx} \tag{5}$$

3.1 Laminar Flow Between Stationary Parallel Plates

Considering the flow of fluid between two parallel fixed plates, the relevant boundary conditions as given below [9-11]. At z = 0 and z = L, U = 0 (the velocity is always zero at the wall). Applying boundary condition at $U(0) = 0$, equation (5) becomes,

$$\mu\left[s^2U(z)-U'(0)\right]=\frac{1}{s}\frac{dP}{dx} \tag{6}$$

In this equation, $U'(0)$ is some constant. So, let substitute $U'(0)=C_1$. Also, since dP/dx is uniform, hence put $dP/dx=-\phi$ where ϕ is a constant and negative sign indicates that the pressure of fluid decreases in the direction of flow of the fluid. By using this relation, equation (3) can be written as,

$$\mu\left[s^2U(z)-C_1\right]=\frac{-\phi}{s} \quad \text{or} \quad U(z)=\frac{C_1}{s^2}-\frac{\phi}{\mu s^3} \tag{7}$$

Taking inverse Laplace transform of equation (7), we get,

$$U(z)=C_1 z-\frac{\phi}{2\mu}z^2 \tag{8}$$

Determination of the Constant, C_1

The value of constant, C_1 can be determined by applying boundary condition $U(L)$ = 0, equation (8) provides,

$$0=C_1 L-\frac{\phi}{2\mu}L^2 \quad \text{or} \quad C_1=\frac{\phi}{2\mu}L \tag{9}$$

Substitute the value of from equation (9) in equation (8) to get the following equation.

$$U(z)=\frac{\phi}{2\mu}Lz-\frac{\phi}{2\mu}z^2=\frac{\phi}{2\mu}\left(Lz-z^2\right) \tag{10}$$

Differentiating equation (10) with respect to z, we get,

$$U'(z)=\frac{\phi}{2\mu}(L-2z) \tag{11}$$

For maximum velocity, $U'(z)=0$. Therefore, this will give a result,

$$z=\frac{L}{2} \tag{12}$$

Put the value of z from equation (12) in equation (10), we get,

$$U_{\max}=\frac{\phi}{2\mu}\frac{L^2}{4}=\frac{\phi}{8\mu}L^2 \tag{13}$$

The equations (10) and (12) confirm that the velocity distribution is maximum at the centre between the stationary parallel plates and decreases parabolically with a maximum value at the centre between the fixed parallel plates to a minimum value at the lower fixed plate as well as at the upper fixed plate. The shear stress distribution is determined by the application of Newton's law of viscosity as,

$$\tau(z)=\mu\frac{dU}{dz}$$

Using equation (8), we get,

$$\tau(z)=\frac{\phi}{2}(L-2z) \tag{14}$$

At $z=L/2$ i.e., at the mid of the fixed parallel plates, $\tau\left(\frac{L}{2}\right)=\frac{\phi}{2}\left(L-2\frac{L}{2}\right)=0$ i.e., there is no shear stress even when there is constant pressure gradient.

At $z=0$ i.e., at the surface of the lower fixed plate,

$$\tau(0)=\frac{\phi}{2}L$$

At $z=L$ i.e., at the surface of the upper fixed plate,

$$\tau(L)=-\frac{\phi}{2}L$$

For a particular case, when $\phi=0,\ \tau(z)=0$ i.e., there is no shear stress between the fixed parallel plates if there is no pressure gradient. The equation (14) confirms that the shear stress varies linearly in the presence of constant pressure gradient between the fixed parallel plates with a minimum value at the midway between the fixed parallel plates to a maximum value at the lower fixed plate as well as at the upper fixed plate.

3.2 Laminar Flow Between Parallel Plates with Relative Motion

Considering the flow of fluid between the parallel flat plates such that the lower plate is fixed at z = 0 and upper plate is moving uniformly with velocity U_o relative to the lower fixed plate in the direction of flow of the fluid, we can write the relevant boundary conditions as given below [9-11]. At z = 0, U = 0 and $z = L$, $U = U_o$.

Applying boundary condition at $U(0)$ = 0, equation (5) becomes,

$$\mu\left[s^2U(z)-U'(0)\right]=-\frac{\phi}{s} \tag{15}$$

In this equation, $U'(0)$ is some constant. So, let substitute $U'(0)=C_2$. Hence, equation (15) become,

$$\mu\left[s^2U(z)-C_2\right]=\frac{-\phi}{s} \quad \text{or} \quad U(z)=\frac{C_2}{s^2}-\frac{\phi}{\mu s^3} \tag{16}$$

Taking inverse Laplace transform of equation (17), we get,

$$U(z)=C_2 z-\frac{\phi}{2\mu}z^2 \tag{17}$$

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Determination of the Constant, C_2

The value of constant, C_2 can be determined by applying boundary condition $U(L) = U_o$, equation (17) provides,

$$U_o = C_2 L - \frac{\phi}{2\mu} L^2 \quad \text{or} \quad C_2 = \frac{U_o}{L} + \frac{\phi}{2\mu} L \tag{18}$$

Substitute the value of from equation (18) in equation (17) to get the following equation.

$$\begin{aligned} U(z) &= \left(\frac{U_o}{L} + \frac{\phi}{2\mu} L \right) z - \frac{\phi}{2\mu} z^2 \\ &= \frac{U_o}{L} z + \frac{\phi}{2\mu} \left(Lz - z^2 \right) \end{aligned} \tag{19}$$

Differentiating equation (19) with respect to z, we get,

$$U'(z) = \frac{U_o}{L} + \frac{\phi}{2\mu} (L - 2z) \tag{20}$$

For maximum velocity, $U'(z) = 0$. Therefore, this will give a result,

$$z = \frac{L}{2} - \frac{\mu U_o^{\,2}}{L\phi} \tag{21}$$

Put the value of z from equation (21) in equation (19), we get,

$$U_{\max} = \frac{\mu U_o^{\,2}}{L^2 \phi} \tag{22}$$

The shear stress distribution is determined by the application of Newton's law of viscosity as,

$$\tau(z) = \mu \frac{dU}{dz}$$

Using equation (20), we get,

$$\tau(z) = \frac{\mu U_o}{L} + \frac{\phi}{2} (L - 2z) \tag{23}$$

At $z = L/2$ i.e., at the mid of the fixed parallel plates, $\tau\left(\frac{L}{2}\right) = \frac{\phi}{2}\left(L - 2\frac{L}{2}\right) = 0$ i.e., there is no shear stress even when there is constant pressure gradient.

At $z = L/2$ i.e., at the mid of the flow passage,

$$\tau\left(\frac{L}{2}\right) = \frac{\mu U_o}{L}$$

At $z = L$ i.e., at the surface of the upper fixed plate,

$$\tau(L) = \frac{\mu U_o}{L} - \frac{\phi}{2} L$$

At $z = 0$ i.e., at the surface of the lower fixed plate,

$$\tau(0) = \frac{\mu U_o}{L} + \frac{\phi}{2} L$$

For a particular case, when $\phi = 0$, $\tau(z) = \mu U_o / L$ i.e., the shear stress between the plates is not zero and having a constant value even if there is no pressure gradient. The equation (23) confirms that the shear stress varies linearly in the presence of constant pressure gradient between the parallel plates having relative motion, and at the midway between the parallel plates it is equal to the mean of the values of the shear stresses at the lower fixed plate and the uniformly moving upper plate.

4. Conclusion

In this present study, we have obtained the velocity distribution and the shear stress distribution of a unidirectional laminar flow between stationary parallel plates as well as laminar flow between parallel plates having a relative motion by solving the differential equation describing the flow characteristics of a viscous fluid via Laplace transform method. As a result, this method is both interesting and novel, and it demonstrates how the Laplace transform can be used to obtain the solution (velocity distribution) of a differential equation representing flow characteristics without having to find the general solution. We concluded that in the case of laminar flow with constant pressure gradient between stationary parallel plates, the velocity distribution is maximum at the midway between the parallel plates and decreases parabolically with maximum value at the midway between the parallel plates to a minimum value at the lower fixed plate as well as at the upper fixed plate but the shear stress varies linearly with a minimum value at the midway between the parallel plates to a maximum value at the lower fixed plate as well as at the upper fixed plate. In the case of laminar flow with constant pressure gradient between parallel plates having a relative motion, the velocity distribution is parabolic with a minimum at the lower fixed plate but the shear stress varies linearly and at the midway between the parallel plates having a relative motion it is equal to the mean of the values of the shear stresses at the lower fixed plate and at the uniformly moving upper plate, and having a constant value even if there is no pressure gradient between parallel plates having a relative motion.

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